Financial Derivatives

Making bets without setting foot in a casino

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Contents

1 Module Description

1.1 Learning Objectives

After successfully completing this course, students will understand fundamental types of financial derivatives and their application for investment and hedging purposes. In particular, students will…

- … be familiar with contract specifications and payoffs for basic derivatives such as forwards and options.
- … be able to apply standard pricing results to determine the value of these contracts.
- … be able to develop and analyze trading strategies based on financial derivatives.

1.2 Course Structure

The course consists of weekly lectures with integrated exercise sessions and group presentations. Formal theoretical results and methods presented during the lectures will be illustrated using examples in Excel. In addition, students will be provided with problem sets containing questions and exercises on important concepts developed during the lectures. Solutions to these problem sets will also be made available on StudyNet. Furthermore, selected problems will be discussed during the exercise sessions in class.

In addition, applications of financial derivatives will be illustrated with case studies assigned to student groups. Each group will work on one case study and present their results in class during the second half of the semester (after the semester break).

1.3 Literature

- Hull, J.C. (2022): Options, Futures, and Other Derivatives, 11th Global Edition, Pearson.

1.4 Examination

1.4.1 1. Examination Component (25%)

- Type: Presentation
- Mode: Analog
- Time: Term time
- Location: On Campus
- Grading: Group work, group grade

- Languages: Questions and answers in English.
- Aids: Free choice, with restrictions defined by faculty.

1.4.2 2. Examination Component (75%)

- Type: Analog written examination
- Mode: Analog
- Time: Term time
- Location: On Campus
- Grading: Individual work, individual grade
- Languages: Questions and answers in English.
- Aids: Closed Book, specific models of Texas Instruments TI-30 series allowed, bilingual dictionaries for non-language exams.

2 Introduction to Financial Derivatives

2.1 Overview

Financial derivatives are financial instruments whose value is derived from the value of an underlying asset, group of assets or benchmark. In modern financial markets, they allow participants to hedge risk, speculate on future movements in asset prices, and gain access to markets or assets that might otherwise be inaccessible. This introduction covers the basic concepts and principles of derivatives and looks at specific types such as forwards, futures, options and swaps, each of which has unique characteristics and applications.

2.2 Derivative Types and Their Functions

2.2.1 Forwards and Futures

Forwards and futures are contractual agreements to buy or sell an asset at a predetermined price at a specified time in the future. While both serve similar economic functions, they differ significantly in their operational details:

- Forwards are private agreements traded over-the-counter (OTC) and tailored to the needs of the contracting parties. They are not standardised and involve higher counterparty risk.
- Futures** are exchange-traded, standardised contracts that reduce counterparty risk through daily settlement and the use of clearing houses.

These instruments are primarily used to hedge against changes in the prices of commodities, currencies and financial instruments and for speculative purposes.

2.2.2 Options

Options provide the buyer the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price before or at the expiration date. They can be classified as:

- **European options**, which can be exercised only at maturity.
- **American options**, which can be exercised at any time up to and including the expiration date.

Options are powerful tools for managing financial risk and structuring complex financial trades based on forecasted market movements.

2.2.3 Swaps

Swaps are contracts in which two parties agree to exchange cash flows or other financial instruments over a specified period. Common types of swaps include:

- **Interest rate swaps**, which involve exchanging fixed-rate interest payments for floating-rate payments, and vice versa, usually on a notional principal amount.
- **Currency swaps**, which involve exchanging principal and interest payments in different currencies.

Swaps help in managing exposure to fluctuations in interest rates, exchange rates, and other financial variables.

2.3 Fundamental Principles

2.3.1 Pricing and Valuation

The pricing of derivatives is based on complex mathematical models that account for various factors, including the current market price of the underlying asset, the strike price, the risk-free rate of return, time to expiration, and volatility. Two fundamental models used in derivatives pricing are:

- **Black-Scholes Model**: Used primarily for pricing European options and certain types of American options.
- **Binomial Model**: A versatile method applicable to a broader range of options, including those with complex features like American options.

2.3.2 Market Dynamics and Risk Management

Derivatives are critical in risk management strategies, enabling institutions and individuals to protect against and manage risks. However, their complexity and leverage can also amplify losses, making it essential for participants to understand the products thoroughly and the risks involved.

3 Forwards and Futures

Forwards and futures are financial derivatives that allow parties to agree to trade an asset at a specified future time and price. Although similar in purpose, forwards and futures differ in terms of trading venues, standardisation, and settlement mechanisms.

3.1 Contract Specifications

3.1.1 Definitions and Terminology

Forward Contracts: A forward contract is a bilateral agreement to buy or sell an asset at a specified price on a future date. They are privately negotiated and traded over-the-counter (OTC), providing customisation but exposing parties to credit risk. The price is determined such that no future payments are needed when the contract is concluded.

Futures Contracts: Similar to forwards, futures are agreements to buy or sell an asset at a predetermined price and future date. However, they are standardised and traded on exchanges, ensuring liquidity and reducing counterparty risk through clearinghouses. A clearinghouse sits in between of both parties and makes sure that the transaction goes smoothly. Famous exchanges include Chicago Mercantile Exchange (CME), Intercontinental Exchange (ICE), and EUREX. Exchanges standardise the following aspects of futures contracts:

1. **Underlying Asset**: Nature and quality, sometimes with price differentials.
2. **Contract Size**: Number of units per contract.
3. **Place of Delivery**: Typically licensed warehouses.
4. **Delivery Months**: Predefined months for delivery.
5. **Other Features**: Position limits, price limits, and settlement methods.

Key Positions:

- **Long Position**: The party agreeing to buy the asset at the delivery price.
- **Short Position**: The party agreeing to sell the asset at the delivery price.

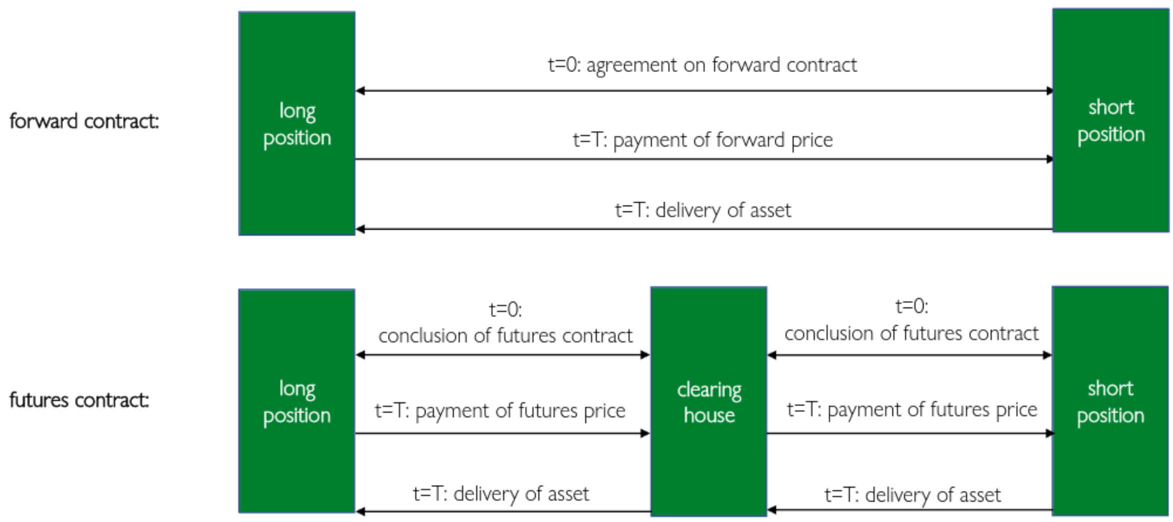

Figure 1: Forwards vs. Futures

Quotes and Terminology:

- **Open**: Price after trading starts.
- **High/Low**: Highest/lowest price during a trading day.
- **Settlement/Last/Close**: Price before trading ends.
- **Volume**: Number of contracts traded.
- **Open Interest**: Number of open positions (either long or short).

3.1.2 Notation and Value at Delivery

Notation:

- S_t: Spot price of the underlying asset at time t.
- T: Contract maturity date or time of delivery.
- $F_t = F(t, T)$: Forward/Futures price at time t for a contract maturing at T.

Contract Value at Delivery:

- For a long position, the value at maturity ($t = T$) is:

$$V_{\text{long}} = S_T - F_0$$

  where S_T is the spot price at maturity and F_0 is the agreed delivery price.

- For a short position, the value at maturity is:

$$V_{\text{short}} = F_0 - S_T$$

3.1.3 Closing Out Positions

Although physical delivery plays an important role in determining futures prices, most contracts are closed out before maturity to avoid physical delivery. Most participants in the futures market are speculators or hedgers who do not wish to actually receive or dispatch the physical good. Clousing out positions is achieved by taking an opposite position, effectively offsetting the initial contract.

However, even when taking the opposite position, the agreements to buy or sell are still in tact. This is were cash settlement comes to play. Cash settlement is a method used in futures contracts where, instead of physical delivery, the seller of the futures contract pays the buyer the cash value of the commodity or asset at maturity. If physical delivery is impractical, cash settlement may be used (e.g., for index futures).

3.1.4 Price Convergence

At maturity, the futures price ($F(T,T)$) must converge to the spot price ($S(T)$). If a discrepancy arises, arbitrage strategies ensure convergence:

- If $F(T,T) < S(T)$:
  - Take a long position in futures.
  - Buy the asset at the futures price and sell at the spot price.
  - **Profit**: $S(T) - F(T,T)$.
- If $F(T,T) > S(T)$:
  - Take a short position in futures.
  - Buy the asset at the spot price and sell at the futures price.
  - **Profit**: $F(T,T) - S(T)$.

3.2 Daily Settlement and Margins

3.2.1 Daily Settlement

Futures contracts differ from forwards in their daily settlement mechanism known as *mark-to-market*. Each trading day, gains and losses are realised and settled in cash.

- **Long Position**: Receives (or pays) $F_t - F_{t-1}$ at the end of day t.
- **Short Position**: Receives (or pays) $F_{t-1} - F_t$ at the end of day t.

Total Profit/Loss at Maturity:

- For a long position:
$$\sum_{t=1}^{T}(F_t - F_{t-1}) = F_T - F_0 = S_T - F_0$$
- For a short position:
$$\sum_{t=1}^{T}(F_{t-1} - F_t) = F_0 - F_T = F_0 - S_T$$

Example: Assume a trader takes a long position in 5 futures contracts at $F_0 = 250$ and closes at $F_5 = 252$. The following table shows the daily profit and cumulative total profit:

Day	Price	Change	Daily Long P&L	Total Long P&L	Daily Short P&L	Total Short P&L
0	250	-	-	-	-	-
1	253	3	1,500	1,500	-1,500	-1,500
2	255	2	1,000	2,500	-1,000	-2,500
3	251	-4	-2,000	500	2,000	-500
4	248	-3	-1,500	-1,000	1,500	1,000
5	252	4	2,000	1,000	-2,000	-1,000

3.2.2 Margin Accounts

Trading futures requires participants to maintain margin accounts to cover potential losses. The margin account is also used to settle daily gains and losses. Additionally, interest is usually paid on the funds in the margin account.

- **Initial Margin**: Deposit required to open a long or short position.
- **Maintenance Margin**: Minimum balance required in the margin account; if it falls below this level, a margin call is triggered.
- **Excess Funds**: Withdrawable if the margin account balance exceeds the initial margin.

Example (Continued): Assume the initial margin is 500 per contract and the maintenance margin is 400 per contract. Here's how the margin account balance evolves:

Day	Price	Change	Daily P&L	Cumulative P&L	Balance	Margin Call	New Balance
0	250	-	-	-	2,500	-	2,500
1	253	3	1,500	1,500	4,000	-	4,000
2	255	2	1,000	2,500	5,000	-	5,000
3	251	-4	-2,000	500	3,000	-	3,000
4	248	-3	-1,500	-1,000	1,500	1,000	2,500
5	252	4	2,000	1,000	4,500	-	4,500

3.3 Hedging Strategies

3.3.1 Short Hedge

A short hedge involves reducing the risk of a decline in the value of an owned asset by taking a short position in futures. This strategy is appropriate if the hedger already owns the asset and plans to sell it in the future:

- Farmers who own livestock and plan to sell it in the future.
- Oil producers who will sell a given number of barrels in the future.
- Portfolio managers who want to protect their portfolios from price drops.

Example: Consider a company holding a long position in a stock with $S_0 = 80$ and taking a short futures position at $F(0, T) = 100$. At maturity:

- Long stock position profit: $S_T - 80$.
- Short futures profit: $100 - S_T$.

Total Profit:

$$S_T - 80 + 100 - S_T = 100 - 80 = 20$$

This is further illustrated by the P&L graph:

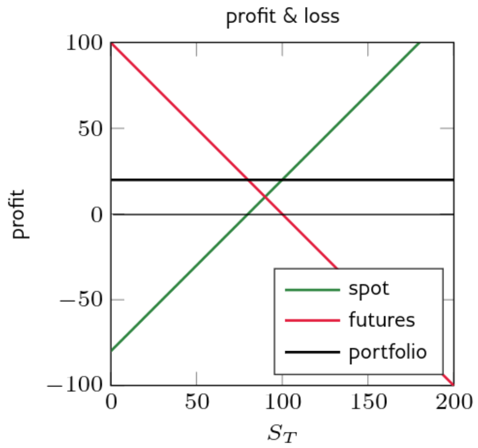

Figure 2: Short Hedge P&L

3.3.2 Long Hedge

A long hedge involves reducing the risk of a rise in the value of a needed asset by taking a long position in futures. This strategy is appropriate for:

- Companies needing specific commodities in the future.
- Companies locking in future foreign currency payments.

Example: A company requires a commodity at time T and anticipates an uncertain price S_T. The unhedged profit is $120 - S_T$. By taking a long futures position at $F(0, T) = 100$, the final hedged profit is:

$$120 - S_T + S_T - 100 = 120 - 100 = 20$$

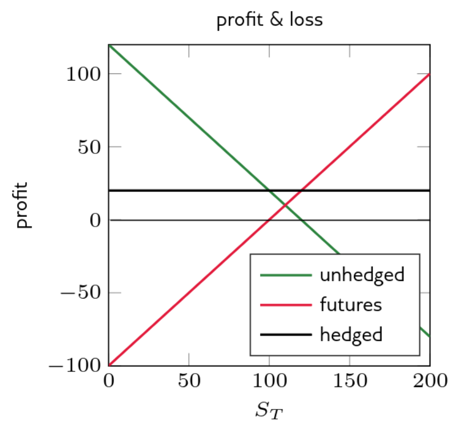

Figure 3: Long Hedge P&L

3.3.3 Reasons for (and Against) Hedging

In Favour:

1. Allows companies to focus on core activities.
2. Hedging costs are lower for companies than for individual investors.
3. Reduces expected bankruptcy costs.
4. Smooths profits over time, potentially reducing taxes.

Against:

1. Does not increase firm value in perfect markets.
2. Shareholders could hedge themselves.
3. Shareholders can diversify to reduce firm-specific risk.
4. Involves transaction costs.

3.3.4 Basis Risk

Perfect hedging aims to eliminate risk entirely, but this is often not feasible due to basis risk. The basis is defined as:

$$\text{Basis} = \text{Price of Asset to be Hedged} - \text{Futures Price of Contract}$$

Sources of Basis Risk:

1. **Cross-Hedging**: When the asset to be hedged differs from the futures contract's underlying.
2. **Maturity Mismatch**: Hedging time horizon differs from the futures contract maturity.

3.3.5 Cross-Hedging

Examples:

- Hedge a stock portfolio with a stock index futures.
- Hedge jet fuel purchases with heating oil futures.

Optimal Hedge Ratio: To minimise the variance of the hedged position:

$$h^* = \text{argmin}_h \, \mathrm{Var}[\Delta S - h\Delta F],$$

where:

- ΔS: Change in spot price over the hedging period.

- ΔF: Change in futures price over the hedging period.

The optimal hedge ratio is:

$$h^* = \frac{\rho_{SF}\sigma_S}{\sigma_F},$$

where:

- ρ_{SF}: Correlation between changes in spot (ΔS) and futures (ΔF) prices.
- σ_S: Standard deviation of spot price changes.
- σ_F: Standard deviation of futures price changes.

Optimal Number of Futures Contracts:

$$N^* = \frac{\text{Units of Asset (Spot Position)}}{\text{Contract Size (Futures)}} \cdot h^*$$

3.4 Forward Pricing

3.4.1 Foundations

Short Selling

Short selling or shorting refers to selling an asset that you do not own. How it typically works:

- $t = 0$: A broker borrows the asset and sells it in the spot market.
- $t = T$: When the short sale is "closed out", the broker buys the asset in the spot market and returns it to the borrower.

Cash Flows: - $+S_0$ at time $t = 0$ from selling the asset. - $-S_T$ at time $t = T$ from buying and replacing the asset.

Profit or Loss: $S_0 - S_T$ (short sellers benefit from falling prices).

Interest Rates and Zero-Coupon Bonds

If the term structure is flat, a single spot interest rate r applies to all maturities. In this case, the present value of a payoff Z occurring at time T is given by:

$$P_0 = Z \cdot e^{-r \cdot T}.$$

In reality, the spot rate typically depends on the maturity of the cash flows. In this case, maturity-specific spot rates $r(T)$ are used for discounting.

- A zero-coupon bond with maturity T and a face value of N is a bond that does not pay coupons. There is only a single payment N at time T. Zero-coupon bonds are used as follows in trading/hedging strategies:
  - Long position in a zero-coupon bond = risk-free investment until time T.
  - Short position in a zero-coupon bond = (risk-free) borrowing until time T.

When pricing assets, we assume that there is no arbitrage opportunities (no free lunch). This is defined in the law of one price (LOOP): 2 portfolios which have the same payoffs in every state of world must have the same price. In asset pricing, we use a stricter definition of absence of arbitrage: Portfolio A must have a higher price than portfolio B if the payoff of A larger equal the payoff of B.

When it comes to the pricing explained below, the following assumptions apply:

- No transaction costs
- No taxes
- Unlimited borrowing or lending at the same rate r
- Market participants try to exploit arbitrage opportunities, creating absence of arbitrage

3.4.2 Forward Price: Investment Asset with No Income

Replication Long Position (No Income)

- **Position 1**: Forward Long

$$\begin{aligned} t = 0: &\quad 0 \\ t = T: &\quad S_T - F_0 \end{aligned}$$

- **Position 2**: Leveraged purchase of the asset in the spot market

$$\begin{aligned} t = 0: &\quad - S_0 + F_0 \cdot e^{-r \cdot T} \\ t = T: &\quad S_T - F_0 \end{aligned}$$

Arbitrage-Free Forward Price \ The payoff of both strategies at $t = 0$ is the same, therefore, the cost of both must be the same

$$\begin{aligned} -S_0 + F_0 \cdot e^{-4 \cdot T} = 0 \\ F_0 = S_0 \cdot e^{r \cdot T}. \end{aligned}$$

3.4.3 Forward Price: Investment Asset with Known Income

Now, let's assume the underlying pays income (interest, dividend) I_τ at time $t = \tau$ with $0 < \tau < T$.

Replication Long Position (With Known Income)

- **Position 1**: Forward Long

$$\begin{aligned} t = 0: &\quad 0 \\ t = \tau &\quad 0 \\ t = T: &\quad S_T - F_0 \end{aligned}$$

- **Position 2**: Leveraged purchase of the asset in the spot market with known income. For that we introduce another zero coupon bond short with face value of I_τ.

$$\begin{aligned} t = 0: &\quad -S_0 + I_\tau \cdot e^{-r\cdot\tau} + F_0 \cdot e^{-r\cdot T} \\ t = \tau: &\quad I_\tau - I_\tau \\ t = T: &\quad S_T - F_0 \end{aligned}$$

Arbitrage-Free Forward Price \ Again, since the payoff at $t = T$ is the same for both portfolios, so must be the price:

$$-S_0 + f_0 \cdot e^{-r\cdot T} + I_\tau \cdot e^{-r\tau} = 0$$
$$F_0 = \left(S_0 - I_\tau \cdot e^{-r\cdot\tau}\right) \cdot e^{r\cdot T}.$$

This means, the present value of the cash income is subtracted from the spot price to adjust the forward contract price, ensuring no arbitrage opportunities arise.

3.4.4 Forward Price: Investment Asset with Known Yield

The average continously compounded yield per annum over the period $[0, T]$ is given by q. The income reinvested in the asset. That means, if we invest in one unit of the asset at time $t = 0$, we get $e^{q\cdot T}$ units at time T.

Replication Long Position (Known Yield)

- **Position 1**: Forward Long

$$\begin{aligned} t = 0: &\quad 0 \\ t = T: &\quad S_T - F_0 \end{aligned}$$

- **Position 2**: Leveraged purchase of the asset in the spot market

$$\begin{aligned} t = 0: &\quad -S_0 \cdot e^{-q\cdot T} + F_0 \cdot e^{-r\cdot T} \\ t = T: &\quad S_T - F_0 \end{aligned}$$

As the asset is paying a yield q, we need less than unit of underlying to get S_T at $t = T$.

Arbitrage-Free Forward Price

$$F_0 = S_0 \cdot e^{(r-q)\cdot T}.$$

3.4.5 Forward Price: Investment Assets with Storage Costs

Now, let's assume fixed storage costs with a present value of U for one unit of the underlying.

Replication (Storage Costs)

- **Position 1**: Forward Long

$$\begin{aligned} t = 0: &\quad 0 \\ t = T: &\quad S_T - F_0 \end{aligned}$$

- **Position 2**: Leveraged purchase of the asset in the spot market with storage costs

$$\begin{aligned} t = 0: &\quad -S_0 - U + F_0 \cdot e^{-r \cdot T} \\ t = T: &\quad S_T - F_0 \end{aligned}$$

Clearly, the forward is cheaper because we do not need to pay for storage.

Arbitrage-Free Forward Price

$$F_0 = (S_0 + U) \cdot e^{r \cdot T}.$$

Storage cost increases the cost of replication and thus the fair forward price. Also, if the storage cost u is proportional to the current price of the asset, they can be understood as a negative yield $q = -u$ and we obtain

$$F_0 = S_0 \cdot e^{(r+u) \cdot T}.$$

3.4.6 Forward Price: Consumption Assets

Replication and Convenience Yield

- **Position 1**: Forward Long

$$\begin{aligned} t = 0: &\quad 0 \\ t = T: &\quad S_T - F_0 \end{aligned}$$

- **Position 2**: Leveraged purchase of the asset in the spot market with consumption benefits

$$\begin{aligned} t = 0: &\quad -S_0 - U + CB + F_0 \cdot e^{-r \cdot T} \\ t = T: &\quad S_T - F_0 \end{aligned}$$

Arbitrage-Free Forward Price

$$F_0 = (S_0 + U - CB) \cdot e^{r \cdot T}.$$

When the exact value of the consumption benefits is unknown, the futures/forward price is bounded by:

$$F_0 \leq (S_0 + U) \cdot e^{r \cdot T}.$$

The convenience yield (y) is implicitly defined as:

$$F_0 \cdot e^{y \cdot T} = (S_0 + U) \cdot e^{r \cdot T}.$$

If y is known, we could solve for F_0:

$$F_0 = S_0 \cdot e^{(r+u-y) \cdot T}.$$

Demand and supply of a commodity typically drive the convenience yield and forms equilibrium pricing.

3.4.7 Additional Results

Cost of Carry

The cost of carry (c) is measured in percent of the current spot price per annum (as are the individual variables) and is defined as:

$$c = r + u - q,$$

where: - r: Financing cost of buying the underlying. - u: Storage cost for the underlying. - q: Income received from the underlying.

Using this, the forward price becomes:

$$F_0 = S_0 \cdot e^{(c-y) \cdot T}.$$

The cost of carry thus captures the difference btween the spot and the forward / futures price.

Value of Forward Contracts

- K: Forward price that was agreed in the past.
- F_0: Fair forward price at time 0.
- T: Time of delivery.
- v_0: Value of a forward contract with delivery price K.

Value Formula at $t = 0$

$$v_0 = (F_0 - K) \cdot e^{-r \cdot T}.$$

This is similar to the daily settlement payment for futures with $K = F_{t-1}$.

Example

Consider a forward contract on a stock with a delivery price $K = 302$. The current stock price is $S_0 = 300$, and the remaining maturity of the contract is 5 months. With a risk-free interest rate of 4.5%:

Calculate Forward Price

$$F_0 = S_0 \cdot e^{0.045 \cdot \frac{5}{12}} = 305.68.$$

Calculate Forward Value

$$v_0 = (F_0 - K) \cdot e^{-0.045 \cdot \frac{5}{12}} = (305.68 - 302) \cdot e^{-0.045 \cdot \frac{5}{12}} = 3.61.$$

3.4.8 Forward vs. Futures Price

There can be a difference between forward and futures prices due to:

1. **Daily Settlement**: Should not generate price differences if interest rates are deterministic.
2. **Delivery Options**: Embedded options in futures contracts can cause price differences. In that case "optimal behaviour" of the option holder should be considered. For example, when delivery period is allowed, assume the earliest delivery if $c > y$ and latest otherwise.
3. **Default Risk**: Differences in default risk can cause price differences.
4. **Other Factors**: Liquidity, taxes, transaction costs, etc.

Despite these factors, it is often reasonable to assume that forward and futures prices are the same in many important cases.

4 Forward Rate Agreements and Swaps

The concepts of Forward Rate Agreements (FRAs) and Swaps are key components used for managing interest rate risks and currency exposures.

4.1 Forward Rates

4.1.1 Definition and Calculation

Forward Rates: These are implied interest rates calculated based on current spot rates for a specified future period.

- **Notation**:
  - $r(t, T)$: Spot rate at time t for the period between t and T.
  - $f(t, T_1, T_2)$: Forward rate at time t that applies for the period from T_1 to T_2 with $t \leq T_1 \leq T_2$.

The forward rate, $f(t, T_1, T_2)$, represents the fixed interest rate agreed upon today for a risk-free investment or loan that will occur between T_1 and T_2. This means the payoff for a forward rate investment is

$$\begin{aligned} t = 0 : &\quad 0 \\ t = T_1 &\quad -L \\ t = T_2 : &\quad L \cdot e^{(T_2 - T_1) \cdot f(t, T_1, T_2)} \end{aligned}$$

Calculation:

The forward rate can be derived by ensuring that the yield from investing at the spot rate over the whole period $[t, T_2]$ is equivalent to investing at the spot rate until T_1 and then reinvesting at the forward rate from T_1 to T_2. The formula for continuous compounding is:

$$f(0, T_1, T_2) = \frac{T_2 \cdot r(0, T_2) - T_1 \cdot r(0, T_1)}{T_2 - T_1}$$

4.1.2 Example

Given:

- 1-year spot rate, $r(0, 1) = 4\%$
- 2-year spot rate, $r(0, 2) = 5\%$

The forward rate for the period from 1 to 2 years would be:

$$f(0, 1, 2) = \frac{2 \cdot 5\% - 1 \cdot 4\%}{2 - 1} = 6\%$$

4.2 Forward Rate Agreements

4.2.1 Definition and Value

Forward Rate Agreements (FRAs): Contracts that fix the interest rate for a future period. Payments are based on a notional amount and do not involve the exchange of principal.

- **Notation**:

- $rK(T_1, T_2)$: Fixed rate specified in the contract.
- $r(T_1, T_2)$: Future variable interest rate for the period $[T_1, T_2]$ at time $t = T_1$. Typically, a reference rate such as overnight rates (SONIA, ESTER, SARON) or repo rates (SOFR) are used.
- L: Notional of the contract.

Cashflows:

- Fixed-rate payer (floating rate receiver): $(r(T_1, T_2) - r_K(T_1, t_2)) \cdot (T_2 - T_1) \cdot L$
- Fixed rate receiver (floating rate payer): $(r_K(T_1, T_2) - r(T_1, T_2)) \cdot (T_2 - T_1) \cdot L$

Valuation:

The value of the FRA to the fixed-rate receiver at time t can be estimated using the difference between the contract rate and the current forward rate, discounted back at the spot rate applicable to T_2:

$$v_t = (rK(T_1, T_2) - f(t, T_1, T_2)) \cdot (T_2 - T_1) \cdot L \cdot (1 + r(t, T_2))^{-(T_2 - t)}$$

4.2.2 Example

If $rK(2, 3) = 4\%$ and the forward rate $f(1, 2, 3) = 5\%$, with $r(1, 3) = 4.5\%$ and a notional of $L = 1,000,000$:

$$v_1 = (5\% - 4\%) \cdot 1 \cdot 1,000,000 \cdot 1.045^{-2} = 9,157.30$$

This means the value of the fixed-rate payer position at time $t = 1$ is $9,157.30$.

4.3 Swaps

4.3.1 Definition and Use

Swaps: Contracts that involve the exchange of payment streams based on different interest rates (or other financial metrics like currency values) over a set period. The interest rate payments are based on the same principal amount.

- **Interest Rate Swap**: Typically involves swapping fixed rate payments for floating rate payments based on the same principal amount. This is the same as a series of FRAs with the same fixed rate.

Example:

A fixed-rate payer agrees to pay 4% annually while receiving a six-month reference rate. Payments are adjusted based on the difference between the agreed rates at fixed intervals. The notional of the conract is $100,000$.

Date	Rate	Cash Flows Fixed	Cash Flows Floating	Net Fixed-rate Payer	Net Floating-rate Receiver
June 2024	4.0%	2,000	2,250	250	-250
Dec 2024	4.0%	2,000	2,500	500	-500
June 2025	4.0%	2,000	2,000	0	0
Dec 2025	4.0%	2,000	1,750	-250	+250

Use:

- **Floating to Fixed**: Converts floating rate liabilities to fixed rates to reduce exposure to interest rate fluctuations.
- **Fixed to Floating**: Offers potential savings when floating rates are expected to be lower than the fixed rates initially contracted.
- **Risk Management**:

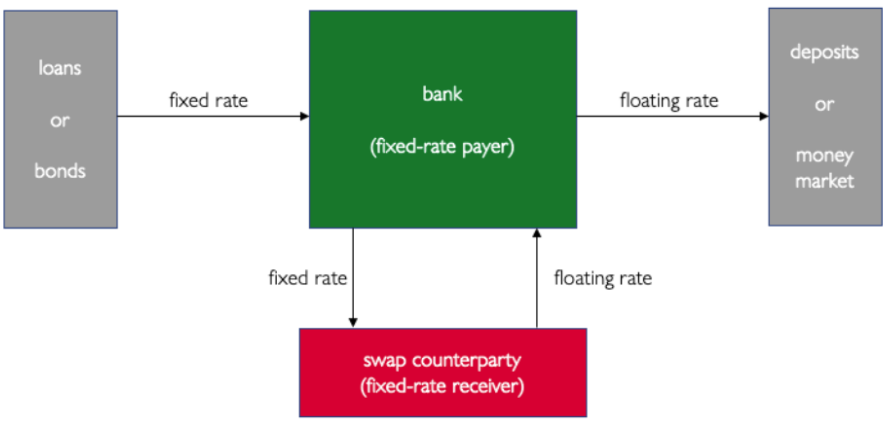

Figure 4: Bank interest rate risk management

Valuation:

Swaps can be seen as a portfolio of forward rate agreements. To determine the current value of a swap, we can apply the following steps:

1. **Determine current forward rates for the payment dates.**

- This involves calculating the expected future interest rates for the periods corresponding to the swap's payment dates.

2. **Calculate payments for each date assuming that the future floating rates correspond to current forward rates.**
    - Project the floating rate payments based on the forward rates determined in the previous step.
3. **Compute the present value of the net payments from step 2 using current spot rates.**
    - Discount the net payments (difference between fixed and projected floating payments) back to the present using the spot rates applicable for each payment period.

Initially, the fixed swap rate is usually chosen such that the value of the contract is zero.

5 Options

Options are versatile financial instruments used for hedging, speculation, and leveraging investments. This section covers the fundamental aspects of options, including their types, pricing bounds, and strategic uses.

5.1 Contract Specifications and Trading

5.1.1 Definitions

Options give the buyer the right, but not the obligation, to buy (call) or sell (put) an underlying asset at a specified strike price K on or before a set expiration date T.

- **Call Option**: Right to buy the asset.
- **Put Option**: Right to sell the asset.
- **American Option**: Can be exercised at any time up to and including the expiration date.
- **European Option**: Can only be exercised precisely at the expiration date.

5.1.2 Notation

- S_t: Current price of the underlying asset at time t.
- K: Strike price of the option.
- T: Expiration date of the option.
- $\tau = T - t$: Time to expiry.

- r: Risk-free interest rate, continuously compounded.
- C_t, P_t: Value of European call and put options at time t.
- $\tilde{C}_t, \tilde{P}_t$: Value of American call and put options at time t.

5.1.3 Payoff Profiles

- **European Call Option**: $C_T = \max(S_T - K, 0)$
- **European Put Option**: $P_T = \max(K - S_T, 0)$

These values represent the intrinsic value, determined at expiration, of the respective options.

You could still exercise the option even if the payoff is < 0 (the option is out of the money), however, you could buy / sell the underlying asset and gain more. Therefore, we do not exercise the option in those cases. In all other cases, i.e. when the option is in the money (positive payoff) or at the money ($S_T = K$), it is worth exercising the option.

The buyer of an option pays a premium to the seller. The profit from buying or selling the option is thus different:

- **long call**: $C_T - C_0 = max(S_T - K, 0) - C_0$
- **short call**: $C_0 - C_T = C_0 - max(S_T - K, 0)$
- **long put**: $P_T - P_0 = max(K - S_T, 0) - P_0$
- **short put**: $P_0 - P_T = P_0 - max(K - S_T, 0)$

5.1.4 Trading Venues

Options are traded on exchanges like the Chicago Board of Options Exchange (CBOE), with standardisation in expiration dates and strike prices, or over-the-counter (OTC), where terms can be customized.

5.2 Price Bounds

Options pricing is influenced by assumptions about the dynamics of the underlying asset. However, certain model-free price bounds can be established through no-arbitrage arguments, especially for European options.

For this section we assume that

- there are no transaction cost,
- all trading profits are subject to the same tax rate,
- borrowing and lending is possible at the risk-free rate,

- the risk-free rate is assumed to be non-negative,
- there is a sufficient number of arbitrageurs that try to exploit arbitrage opportunities.

5.2.1 Upper Bounds

- **Call Option**: Cannot exceed the current stock price, $C_0 \leq S_0$.
- **Put Option**: Cannot exceed the discounted strike price, $P_0 \leq K \cdot \exp(-r \cdot T)$.

American options are at least as valuable as European options, i.e., $P_0 \leq \tilde{P}_0$ and $C_0 \leq \tilde{C}_0$. Additionally, American put options have a higher upper bound due to early exercise.

5.2.2 Lower Bounds

The lower bounds are given by the risk-free rate:

- **Call Option**: $C_0 \geq \max(S_0 - K \cdot \exp(-r \cdot T), 0)$
- **Put Option**: $P_0 \geq \max(K \cdot \exp(-r \cdot T) - S_0, 0)$

5.2.3 Put-Call Parity

Establishes a relationship between the prices of calls and puts of the same strike price and maturity:

$$P_0 + S_0 = C_0 + K \cdot \exp(-r \cdot T)$$

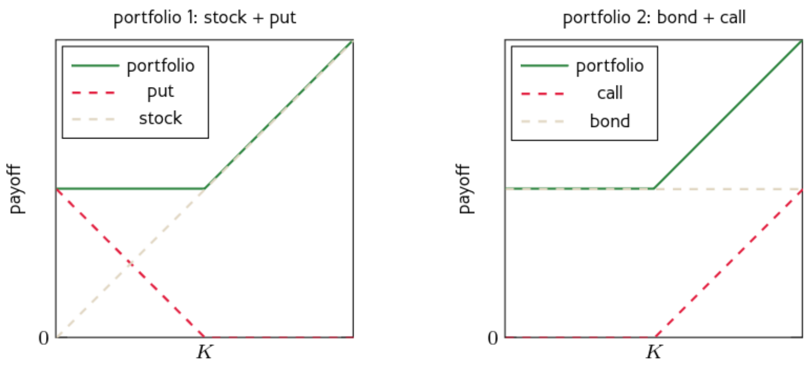

Figure 5: Put-Call parity

This equation helps determine one option's price if the other's price is known, facilitating arbitrage opportunities if the relationship does not hold.

For American options, similar inequalities can be derived.

5.3 Trading Strategies

5.3.1 Protective Put

Involves buying a stock and a put option, providing downside protection while allowing for unlimited upside potential. The cost of this strategy is the premium paid for the put.

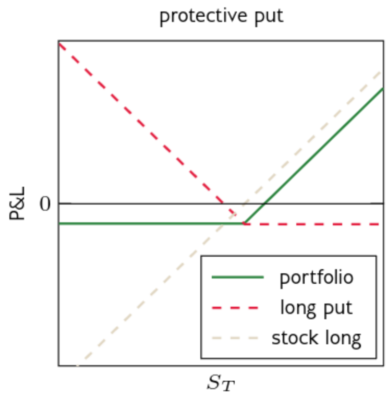

Figure 6: Protective put

The same payoff profile can be created using a zero-coupon bond and a call option according to the put-call parity.

5.3.2 Covered Call

Involves owning a stock and selling a call option against it. This strategy generates income through the option premium, providing a buffer against losses but limiting upside potential if the stock price rises significantly.

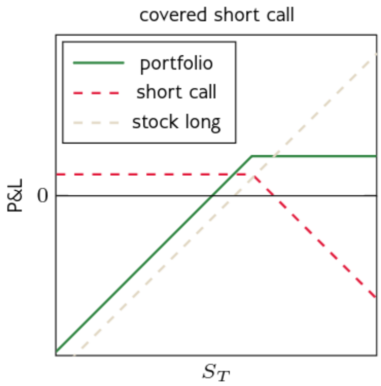

Figure 7: Covered short call

5.3.3 Other Strategies

- **Spreads**: Combining several options of the same type to limit risk and potential return.
- **Combinations**: Strategies like straddles or strangles that involve buying or selling calls and puts together, betting on volatility rather than direction.

These options strategies allow investors to tailor their exposure to risk and reward, depending on their market outlook and risk tolerance.

6 Options – Binomial Model

The Binomial Model provides a discrete and straightforward framework for option pricing. It allows analyzing the potential movements in the price of the underlying asset.

6.1 One-Step Model

6.1.1 Notation and Assumptions

The one-step binomial model involves the following key elements:

- S_t: Price of a non-dividend paying stock at time t.
- X_t: Price of a derivative (such as a call or put option) on the stock at time t.
- r: Risk-free rate, continuously compounded.

The stock price at time T is assumed to move to either $S_u = S_0 \cdot u$ with probability p (where $u > 1$) or $S_d = S_0 \cdot d$ with probability $1 - p$ (where $d < 1$).

As previously, we assume the following:

- No transaction costs
- No taxes (or equal taxes for all profits)
- Unlimited borrowing or lending at the risk-free rate
- Market participants exploit arbitrage opportunities

6.1.2 Example: Call and Put Options

6.1.2.1 Call Option Example For a stock priced at $S_0 = 100$, the price may increase by 15% or decrease by 10% over the period, with a strike price (K) of 110:

- If $S_T = 115$, $C_T = \max(115 - 110, 0) = 5$.
- If $S_T = 90$, $C_T = \max(90 - 110, 0) = 0$.

6.1.2.2 Put Option Example Using the same stock price movements with a strike price of 105:

- If $S_T = 115$, $P_T = \max(105 - 115, 0) = 0$.
- If $S_T = 90$, $P_T = \max(105 - 90, 0) = 15$.

6.1.3 Replication and Pricing

The binomial model uses replication to derive the option prices. The goal is to create a portfolio consisting of Δ shares of stock and a risk-free bond that replicates the payoff of the option:

1. **Determine Δ**: The number of shares needed to hedge the option.
2. **Calculate the bond investment (y)**: The amount in zero-coupon bonds needed to finance the hedge with maturity T.

At time T, it must hold that

$$\Delta \cdot S_t + y \cdot e^{r \cdot T} = X_T.$$

Now plugging in the price changes, we obtain the following conditions:

$$\Delta \cdot S_0 \cdot u + y \cdot e^{r \cdot T} = x_u$$
$$\Delta \cdot S_0 \cdot d + y \cdot e^{r \cdot T} = x_d.$$

By subtracting we get

$$\Delta \cdot S_0 \cdot (u - d) = x_u - x_d,$$

which implies that

$$\Delta = \frac{x_u - x_d}{S_0 u - S_0 d}$$
$$y = (x_u - \Delta \cdot S_0 \cdot u) \cdot e^{-r \cdot T}.$$

The price of the option at time $t = 0$, X_0, is the cost of setting up this hedging portfolio. From the law of one price, we conclude

$$X_0 = \Delta \cdot S_0 + y$$
$$= \delta \cdot S_0 + (x_u - \Delta \cdot S_0 \cdot u) \cdot e^{-r \cdot T}.$$

Another approach is detailed on slide 15 onwards. However, in both cases, the physical probabilities of the underlying moving up or down are irrelevant for the two pricing techniques.

6.2 Risk-Neutral Valuation

A third approach to pricing options is risk-neutral valulation. Using this approach, we determine the expected value of the option payoff at time T in the so-called "risk-neutral world" and discount it with the risk-free rate.

6.2.1 Theory and Application

Risk-neutral valuation calculates the expected payoff of the option under the assumption that all investors are risk-neutral, meaning they do not require additional compensation for risk:

1. **Determine risk-neutral probabilities**: Adjust the probabilities to reflect a risk-free environment.
2. **Calculate the expected payoff**: Compute the expected payoff of the option using these adjusted probabilities.

This is not true in the real world, where investors are risk-averse and thus require compensation for risk (risk premia). Nevertheless, we can determine the real-world prices of derivatives assuming a risk-neutral world.

Risk aversion does not affect derivative pricing because the derivative's value is derived from the underlying asset whose price already incorporates market risk preferences, and can be replicated exactly through dynamic trading in a risk-neutral framework.

The option's current price is the present value of this expected payoff, discounted at the risk-free rate.

6.2.2 Calculation

Risk neutral probabilities are referred to as probabilities under the "risk-neutral probability measure" Q. Expectations under this measure are written as $\mathbb{E}_Q$. The calculation works as follows:

1. Determine the risk-neutral probabilities from prices of the underlying, that means, determine Q such that
$$S_0 = \mathbb{E}_Q[S_t] \cdot e^{-r \cdot T}.$$
2. Calculate the price of derivatives as discounted expected payoffs under the risk-neutral measure, i.e., using the risk-neutral probabilities from step 1, that means,
$$X_0 = \mathbb{E}_Q[X_t] \cdot e^{-r \cdot T}.$$

In the one-step binomial model, the risk-neutral distribution is determined by the probability that the stock price increases, which we denote by q. Using this, the equation becomes

$$S_0 = \mathbb{E}[S_T] \cdot e^{-r \cdot T} = (S_0 \cdot u \cdot q + S_0 \cdot d \cdot (1-q)) \cdot e^{-r \cdot T}.$$

By solving for q, we conclude
$$q = \frac{e^{r \cdot T} - d}{u - d}.$$
Using q and the payoffs x_u and x_d, we can compute

$$X_0 = \mathbb{E}_Q[X_T] \cdot e^{-r \cdot T} = (q \cdot x_u + (1-q) \cdot x_d) \cdot e^{-r \cdot T}.$$

Two-Step Model

6.2.3 Extension of the One-Step Model

The two-step model extends the binomial approach to two periods, allowing for more paths and more precise modeling of price dynamics over time.

6.2.4 Process

1. **Price at each final node**: Compute the option's payoff at each final node of the binomial tree.

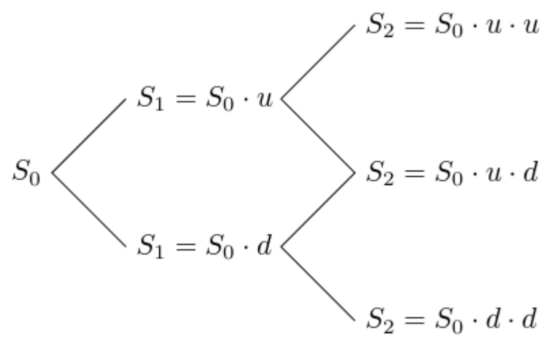

Figure 8: Two-step stock price calculation

2. **Backward induction**: Work backward from the final payoffs to determine the option's value at earlier nodes, including the initial price.

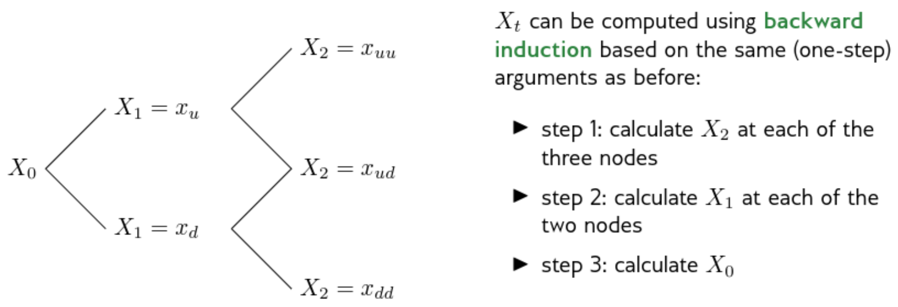

Figure 9: Two-step option pricing calculation

6.3 Multi-Step Models

The step method can be applied with arbitrary number of steps N (in practice values of $N \geq 30$ are used). These binomial trees can easily be used to price all kinds of exotic options.

To transfer this model for use with American options, we need to check at each node wheter early exercise is better for the option holder.

7 Options – Black-Scholes Model

7.1 Background and Assumptions

The Black-Scholes Model is a framework to determine the fair value of European options.

7.1.1 Assumptions

The Black-Scholes Model is built upon the following assumptions:

- **Continuous Trading**: It assumes that the price of the underlying asset changes continuously over time and trading can occur at any instant.
- **No Arbitrage**: Markets are frictionless with no arbitrage opportunities. This means there are no transaction costs or taxes, and market participants can borrow or lend at a constant risk-free rate.
- **Risk-Free Rate**: The risk-free rate (r) is constant and continuously compounded.
- **Geometric Brownian Motion (GBM)**: The price of the underlying asset follows a GBM, implying a constant drift (μ) and volatility (σ).
- **No Dividends**: The underlying asset does not pay dividends during the life of the option.

7.1.2 Geometric Brownian Motion

The model assumes that the underlying asset price (S_t) follows a GBM, defined by the differential equation:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right],$$

where:

- μ: Drift parameter (expected return) of the asset.
- σ: Volatility parameter of the asset's returns.
- W_t: Wiener process or standard Brownian motion.

This implies

$$lnS_t = lnS_0 + (\mu - \frac{\sigma^2}{2}) \cdot t + \sigma \cdot W_t.$$

lnS_t follows a Brownian Motion with drift $m = \mu - \frac{\sigma^2}{2}$ and volatility σ.

The price of the underlying S_t at a fixed time t follows a log-normal distribution with a location parameter of $lnS_0 + (\mu - \frac{\sigma^2}{2}) \cdot t$ and a dispersion parameter of $\sigma \cdot \sqrt{t}$. Visually, this represents a curve like this:

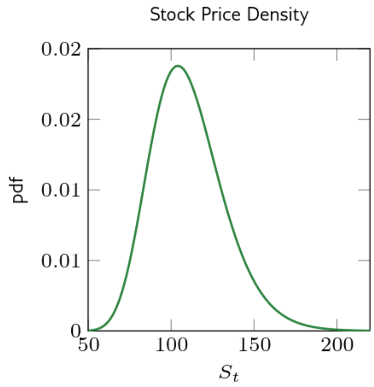

Figure 10: $t = 1, \mu = 10\%, \sigma = 20\%, S_0 = 100$

It is not difficult to show that

$$\mathbb{E}[S_t] = S_0 \cdot e^{\mu \cdot t}$$
$$Var[S_t] = S_0^2 \cdot e^{2 \cdot \mu \cdot t} \cdot (e^{\sigma^2 \cdot t} - 1).$$

The log-return of the underlying over $[0, t]$, i.e., $U_t = lnS_t - lnS_0$, is normally distributed with location parameter $m \cdot t = (\mu - \frac{\sigma^2}{2}) \cdot t$ and volatility parameter $\sigma \cdot \sqrt{t}$. The graph with the same parameters looks as follows:

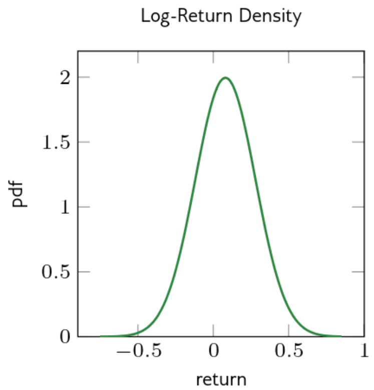

Figure 11: $t = 1, \mu = 10\%, \sigma = 20\%, S_0 = 100$

It holds that

$$\mathbb{E}[U_t] = m \cdot t$$
$$Var[U_t] = \sigma^2 \cdot t$$

The geometric Brownian motion is much more suitable as stock price model, but it's not perfect. It assumes volatility is constant and that there are no sudden jumps.

7.2 Pricing Results

7.2.1 Black-Scholes Formula

The Black-Scholes formula provides a way to calculate the price of European call and put options.

For a European call option, the price (C_t) at time t is given by:

$$C_t = S_t \cdot N(d_1) - K \cdot e^{-r\cdot(T-t)} \cdot N(d_2)$$

For a European put option, the price (P_t) at time t is:

$$P_t = K \cdot e^{-r\cdot(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1),$$

where:

- $d_1 = \frac{\ln(S_t/K)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$
- $d_2 = d_1 - \sigma\sqrt{T-t}$
- $N(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

As we assume that S_t follows a GBM, μ refers to the drift and σ to the volatility.

7.2.2 Interpretation and Insights

- **No Arbitrage and Put-Call Parity**: The formula adheres to the no-arbitrage principle and is consistent with the put-call parity relationship.
- **Absence of Early Exercise**: In the Black-Scholes framework, it is never optimal to exercise a call option early if the underlying asset does not pay dividends. Therefore, it also applies to American call options. However, there is no simple way for American put options using the framework.

7.2.3 Example Calculations

Given the parameters $S_0 = 100$, $K = 105$, $r = 4\%$, $\mu = 10\%$, $\sigma = 20\%$, and $T = 0.5$ (6 months):

- Calculate d_1 and d_2:

$$d_1 = \frac{\ln(100/105) + (0.04 + 0.2^2/2) \cdot 0.5}{0.2\sqrt{0.5}} = -0.1329$$

$$d_2 = d_1 - 0.2\sqrt{0.5} = -0.2743$$

- Call option price:

$$C_0 = 100 \cdot N(-0.1329) - 105 \cdot e^{-0.04 \cdot 0.5} \cdot N(-0.2743) = 4.38$$

- Put option price:

$$P_0 = 105 \cdot e^{-0.04 \cdot 0.5} \cdot N(0.2743) - 100 \cdot N(0.1329) = 7.30$$

If, instead of $S_0 = 100$, we model the price according to S_0, we get the following price graphs:

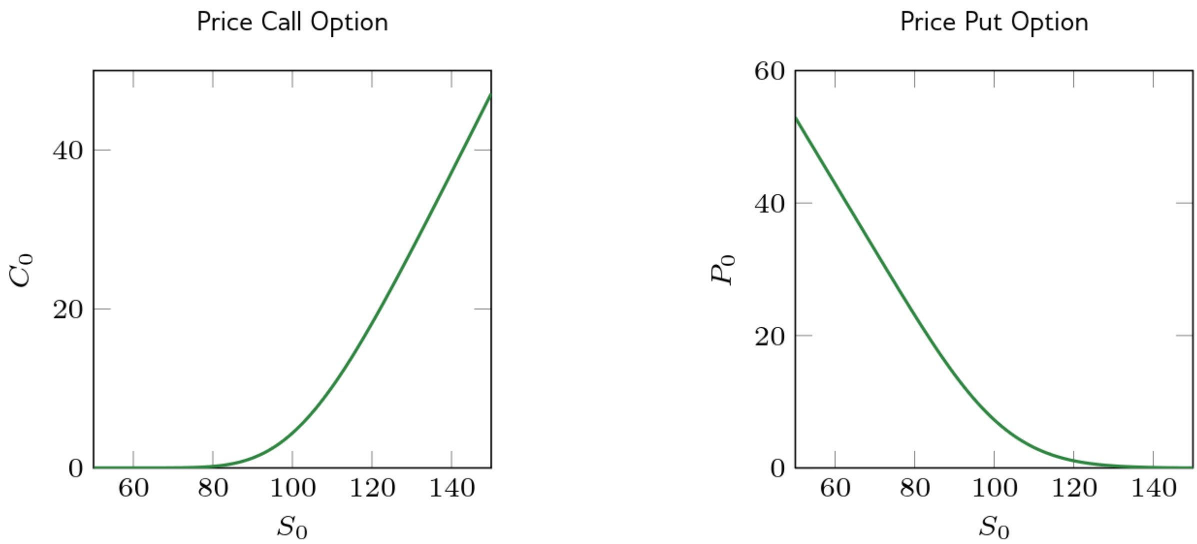

Figure 12: Black-Scholes prices as functions of the current price of the underlying

7.3 The Greek Letters

The Greek letters (or Greeks) are used to measure the sensitivity of the option price to various factors. They provide insights for risk management and hedging strategies.

7.3.1 Delta (△)

Delta measures the sensitivity of the option price to changes in the price of the underlying asset.

- For a call option: $\Delta_C = \frac{\partial C_t}{\partial S_t} = N(d_1) \geq 0$
- For a put option: $\Delta_P = \frac{\partial P_t}{\partial S_t} = N(d_1) - 1 \leq 0$

This relationship is demonstrated by these graphs:

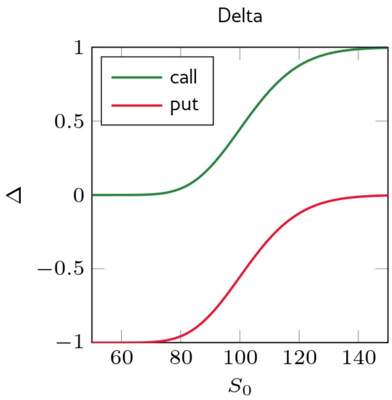

Figure 13: Black-Scholes Δ

7.3.2 Gamma (Γ)

Gamma measures the rate of change in delta with respect to changes in the underlying asset price.

$$\Gamma = \frac{\partial^2 C_t}{\partial S_t^2} = \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}} > 0,$$

where N' denotes the probability density function of the standard normal distribution. The option delta increases with the price of the underlying.

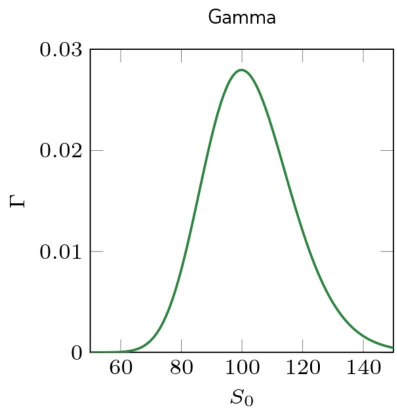

Figure 14: Black-Scholes Γ

7.3.3 Theta (Θ)

Theta measures the sensitivity of the option price to the passage of time.

- For a call option:

$$\Theta_C = -\frac{S_t N'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

- For a put option:

$$\Theta_P = -\frac{S_t N'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

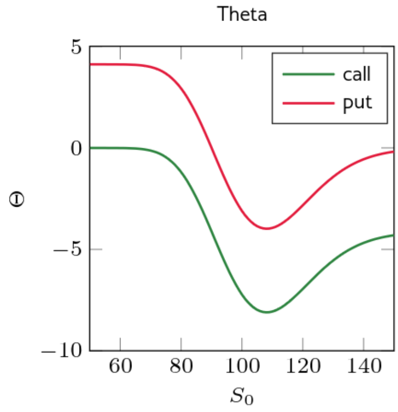

Figure 15: Black-Scholes Θ

7.3.4 Rho (ρ)

Rho measures the sensitivity of the option price to changes in the risk-free interest rate.

- For a call option:

$$\rho_C = K(T-t)e^{-r(T-t)}N(d_2) \geq 0$$

- For a put option:

$$\rho_P = -K(T-t)e^{-r(T-t)}N(-d_2) \leq 0$$

The call (put) price increases (decreases) with the level of the risk-free rate:

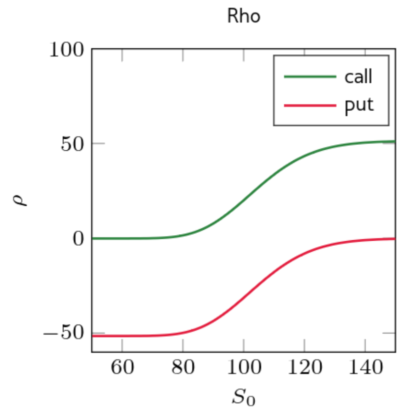

Figure 16: Black-Scholes ρ

7.3.5 Vega (ν)

Vega measures the sensitivity of the option price to changes in the volatility of the underlying asset.

$$\nu = \frac{\partial C_t}{\partial \sigma} = S_t\sqrt{T-t}N'(d_1) \geq 0$$

Option prices increase when the volatility of the underlying increases:

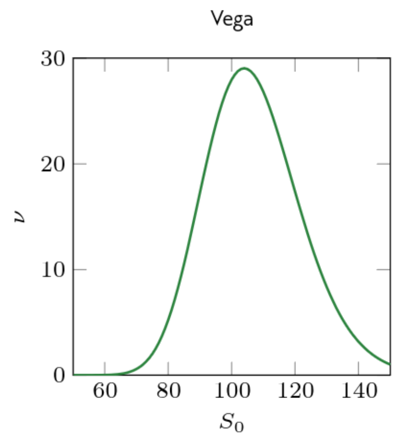

Figure 17: Black-Scholes ν

7.3.6 Example: Greek Values Calculation

Using the same parameters as the previous example ($S_0 = 100, \sigma = 20\%, r = 4\%, K = 105, T = 0.5$), we can compute the Greeks for the call option:

- $\Delta_C = N(-0.1329) = 0.4471$: If the stock price increases by 1, the price of the call option increases by approximately 0.4471.
- $\Gamma = \frac{N'(-0.1329)}{100 \cdot 0.2\sqrt{0.5}} = 0.0280$: If the stock price increases by 1, the delta will increase by approximately 0.028.
- $\Theta_C = -\frac{100 \cdot N'(-0.1329) \cdot 0.2}{2\sqrt{0.5}} - 0.04 \cdot 105 \cdot e^{-0.04 \cdot 0.5} \cdot N(-0.2743) = -7.2058$: The price of the call option decreases by $\frac{-7.2058}{365} = 0.02$ per calendar day or 0.03 per trading day.
- $\rho_C = 105 \cdot 0.5 \cdot e^{-0.04 \cdot 0.5} \cdot N(-0.2743) = 20.1690$: A 1 percentage point increase, makes the call price increase by $20.169 \cdot 0.01 = 0.2$.
- $\nu = 100 \cdot \sqrt{0.5} \cdot N'(-0.1329) = 27.9616$: An increase of 1 percentage point of the volatility causes the call price to increase by $27.9616 \cdot 0.01 = 0.28$.